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GENERALIZATION OF CONTINUOUS POSETS

BY

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ABSTRACT. In this paper we develop a general theory of continuity in partially ordered sets. Among the interesting special cases of this theory is the theory of continuous lattices developed by D. Scott, J. Lawson and others.

Introduction. In this paper we are going to generalize the concept of a continuous poset, using the theory of Galois connections. In this sense we are following Hofmann and Stralka who in [**8**] first used this technique in connection with continuous lattices.

In [**19**] Wagner, Thatcher and Wright developed a "uniform approach" method which enable them to generalize the concept of an algebraic poset. At the end of that article they suggested that continuous posets might be investigated by the same method. This paper is a response to their suggestion.

An essential idea of this article is based on the following two facts.

(1) A lattice (or poset) P is a complete lattice if and only if the function $x \to \downarrow x$ from P to the ideal completion of P has a right adjoint sending the ideal I to $\sup(I)$.

(2) This right adjoint has itself a right adjoint (sending x to $\Downarrow x$) if and only if the lattice (or poset) P is continuous. We use the existence of these three functions (for arbitrary extensions of posets) as our definition of continuity.

§1 introduces the main tool which we use in this thesis, the theory of Galois connections. In §2 we present the main concept, that of a *continuous extension*, and develop some basic results concerning it. In particular, we show that a continuous extension naturally gives rise to a dually residuated closure operator (which, in turn, determines the continuous extension up to isomorphism). We also show that in some special cases the concepts of continuity and distributivity coincide. Some results in this part of the article overlap with the work done in [**1, 6**]. The third section is devoted to a general study of well below relations which helps us to construct examples of continuous extensions. This part is closely related to the section about auxiliary relations in [**7**].

The next section is mainly devoted to the construction of two operators associated with a given continuous extension. One of these is a closure operator, the other is an anticlosure operator. One of the main theorems of this paper is Theorem 4.18 which leads the reader to the notion of *strong continuity*. We show that a continuous poset

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(in the original sense of Scott and Markowsky, whose concept is linked to that of a directed set) is always strongly continuous. This provides further insight into the role of the interpolation property in continuous posets. In Theorem 4.18 we prove the equivalence of the interpolation property to other properties, in particular to the *one-step closure property* and *one-step anti-closure property*.

Next we investigate some implications of strong continuity. This permits us to prove some new results in the theory of continuous posets (Corollaries 5.7 and 5.10).

The last section is devoted to the theory of *algebraic extensions*.

1. Basic concepts. In this part we shall introduce some basic notations and defintions concerning posets and some of the basic theory of Galois connections. For a more detailed treatment we refer the reader to [**7**].

Notation. Let $f\colon X \to Y$ be a partial map between the sets X and Y. Let $A \subseteq X$ and $B \subseteq Y$. Denote by f/A the restriction of f to A and by $B//f$ the corestriction of f to B. If $B = \mathrm{Im}(f)$, $i_f\colon B \to Y$ is the inclusion map of B in Y, and j_f is the map $\mathrm{Im}(f)/f$, then we have $i_f \circ j_f = f$.

Notation. Let P be a poset and let $x \in P$. We define

$$\downarrow x = \{y \in P \mid y \leqslant x\} \qquad \text{and} \qquad \uparrow x = \{y \in P \mid y \geqslant x\}.$$

The sets

$$\uparrow S = \bigcup \{\downarrow x \mid x \in S\} \qquad \text{and} \qquad \uparrow S = \bigcup \{\uparrow x \mid x \in S\}$$

are called the *lower set* and the *upper set generated by S*, respectively.

We use P^* to denote the *dual* of the poset P. We use $L(P)$ to denote the poset consisting of the lower subsets of P, ordered by inclusion.

1.1 DEFINITION. Let P be a poset. By a *system of subsets of P* we mean a family M of subsets of P such that every principal ideal of P is $\downarrow S$ for some $S \in M$.

Important examples of systems of subsets are

$\mathfrak{D}$ $(= \mathfrak{D}(P))$, the family of all directed subsets of P,

$\mathfrak{F}$ $(= \mathfrak{F}(P))$, the family of all finite subsets of P, and

$\mathfrak{P}$ $(= \mathfrak{P}(P))$, the family of all subsets of P.

1.2 DEFINITION. Let P be a poset. For each system M of subsets of P, we define $I_M(P) = \{\downarrow S \mid S \in M\}$. $I_M(P)$ (ordered by inclusion) is called *the poset of all M-generated lower sets of P*. The map $a_M\colon P \to I_M(P)$ given by $a_M(x) = \downarrow x$ for each $x \in P$ is called the *natural imbedding* of P into $I_M(P)$.

1.3 DEFINITION. By a *closure structure* we mean an ordered pair (P, f) such that $f\colon P \to P$ is a closure operator. By an *isomorphism* from a closure structure (P, f) to a closure structure (P_1, f_1) we mean a function h such that

(1) h is an order isomorphism from P onto P_1 and

(2) $h \circ f = f_1 \circ h$.

Anti-closure structures and *isomorphisms of anti-closure structures* are defined dually. Anti-closure operators are also known as *kernel operators*.

Hofmann and Stralka [**8**] pointed out the usefulness of the concept of Galois connection in the theory of continuous lattices.

We also will use this concept as a main tool in our theory of continuous extensions. The definitions and lemmas in this section will help to fix the terminology. (For proofs, see [**7** or **8**].)

1.4 DEFINITION. Let P and Q be posets and let f: $P \to Q$ be an order preserving map. The map f is called *residuated* (*dually residuated*) if for each $y \in Q$ there exists $x \in P$ such that $f^{-1}(\downarrow y) = \downarrow x$ ($f^{-1}(\uparrow y) = \uparrow x$).

1.5 LEMMA. *Let P and Q be posets and f: $P \to Q$ an order preserving map. The map f is residuated if and only if for every $y \in Q$ the set $\{u \in P \mid f(u) \leqslant y\}$ has a greatest element.*

Dually, f is dually residuated if and only if for every $y \in Q$ the set $\{u \in P \mid f(u) \geqslant y\}$ has a least element.

1.6 REMARK. Residuated (dually residuated) maps preserve all existing joins (meets). Moreover, if P is a complete lattice and f: $P \to Q$ preserves all existing joins (meets), then f is residuated (dually residuated).

1.7 DEFINITION. With each residuated (dually residuated) map f: $P \to Q$ we associate a corresponding map g: $Q \to P$ called the *left adjoint* (*right adjoint*) of f. For each $y \in Q$, $g(y)$ is the element $x \in P$ such that $f^{-1}(\downarrow y) = \downarrow x$ ($f^{-1}(\uparrow y) = \uparrow x$).

1.8 DEFINITION. Let P, Q be posets. A pair (g, f) of order preserving maps f: $P \to Q$, g: $Q \to P$ is called a *Galois connection* from Q to P if for all $x \in P$ and all $y \in Q$, $x \leqslant g(f(x))$ and $f(g(y)) \leqslant y$.

1.9 LEMMA. *Let f: $P \to Q$ and g: $Q \to P$ be a pair of order preserving maps. The following facts are equivalent.*

(1) *(g, f) is a Galois connection from Q to P.*

(2) *For all $x \in P$ and all $y \in Q$, $x \leqslant g(y)$ if and only if $f(x) \leqslant y$.*

(3) *f is residuated and g is the left adjoint of f.*

(4) *g is dually residuated and f is the right adjoint of g.*

(5) *(f, g) is a Galois connection from P^* to Q^*.*

1.10 LEMMA. *Let Q, P and L be posets. If (g, f) is a Galois connection from Q to P, and (g_1, f_1) is a Galois connecton from P to L, then $(g_1 \circ g, f \circ f_1)$ is a Galois connection from Q to L.*

1.11 REMARK. By Lemma 1.5, one member of a Galois connection uniquely determines the other one.

1.12 LEMMA. *If (g, f) is a Galois connection from Q to P, then for all $x \in Q$ and all $y \in P$, $g(f(g(x))) = g(x)$ and $f(g(f(y))) = f(y)$.*

1.13 COROLLARY. *If (g, f) is a Galois connection from Q to P, then the following conditions hold.*

(1) *$f(P)$ and $g(Q)$ are order isomorphic.*

(2) *f is onto if and only if g is one-one, and g is one-one if and only if for all $y \in Q$, $f(g(y)) = y$.*

(3) *g is onto if and only if f is one-one, and f is one-one if and only if for all $x \in P$, $g(f(x)) = x$.*

1.14 LEMMA. *If* (g, f) *is a Galois connection from* Q *to* Q, *then* $f^2 = f$ *if and only if* $g^2 = g$.

2. Continuous extensions. In this section we will generalize the concept of a continuous poset.

In **[19]** Wagner, Thatcher and Wright suggested that the theory of continuous posets might be generalized through the use of their "uniform approach" method.

Here we follow this suggestion and even go beyond it, using the notion of a Galois connection in defining the concept of a continuous extension.

2.1 DEFINITION. A poset P will be called *M-complete* if M is a system of subsets of P and $\bigvee S$ exists for every $S \in M$.

Thus, for example, a poset P is $\mathcal{P}$-complete if and only if P is a complete lattice, a poset P is $\mathcal{F}$-complete if and only if P is a join semilattice. It is well known that a poset P is $\mathcal{D}$-complete if and only if P is chain complete (this means that every chain in P has a least upper bound).

2.2 DEFINITION. For each M-complete poset P we define a corresponding binary relation on the set P called the *well below relation modulo M* and denoted by $\ll_M$. For $x, y \in P$, $x \ll_M y$ holds if and only if, for each $S \in M$, if $y \leqslant \bigvee S$, then there exists $s \in S$ such that $x \leqslant s$.

2.3 DEFINITION. A poset P is *M-continuous* if the following three conditions hold.

(1) P is M-complete.

(2) For every $y \in P$, $\{x \in P \mid x \ll_M y\} \in I_M(P)$.

(3) For every $y \in P$, $\bigvee \{x \in P \mid x \ll_M y\} = y$.

Using the concept of a Galois connection we can obtain a further generalization of the notion of a continuous poset.

2.4 DEFINITION. By an *extension* we mean an ordered triple (P, Q, a) where P and Q are posets and a is an order imbedding of P into Q. By an *isomorphism* from an extension (P, Q, a) to an extension (P_1, Q_1, a_1) we mean an ordered pair of functions (f, g) such that

(1) f is an order isomorphism from P onto P_1.

(2) g is an order isomorphism from Q onto Q_1.

(3) $g \circ a = a_1 \circ f$.

2.5 DEFINITION. By a *complete extension* we mean an extension (P, Q, a) such that the map a is dually residuated.

2.6 LEMMA. *P is M-complete if and only if $(P, I_M(P), a_M)$ is a complete extension. Moreover, if P is M-complete and b_M is the right adjoint of a_M, then for all $S \in M$ we have $g_M(\downarrow S) = \bigvee S$.*

PROOF. (1) Assume that P is M-complete. Define the map b_1: $I_M(P) \to P$ as follows: For each $T \in I_M(P)$, there exists $S \in M$ such that $T = \downarrow S$. Put $b_1(T) = \bigvee S$. (Clearly, $b_1(T)$ does not depend upon the particular S used to satisfy this condition.) We will show that a_M is dually residuated and that b_1 is the right adjoint of a_M. By Lemma 1.9 it suffices to show that for all $T \in I_M(P)$ and all $x \in P$, $b_1(T) \leqslant x$ if and only if $T \subseteq a_M(x)$. By Definition 1.2 we can translate this to mean: For all $S \in M$ and all $x \in P$, $\bigvee S \leqslant x$ if and only if $S \subseteq \downarrow x$. This equivalence follows from the definition of least upper bound.

By Lemma 1.9 a_M is dually residuated. Thus $(P, I_M(P), a_M)$ is a complete extension and $b_1 = b_M$.

(2) Assume that $(P, I_M(P), a_M)$ is a complete extension. To prove that P is M-complete let $S \in M$ and define $T \in I_M(P)$ by $T = \downarrow S$. Since, by assumption, a_M is dually residuated, there exists $x \in P$ such that $a_M^{-1}(\uparrow T) = \uparrow x$. By Definition 1.2 we see that for $p \in P$, $p \in a_M^{-1}(\uparrow T)$ holds if and only if p is an upper bound for T. It now follows that $x = \bigvee T$, and thus also that $x = \bigvee S$. This shows that P is M-complete. By the result obtained in (1), $b_M(\downarrow S) = \bigvee S$ for each $S \in M$.

2.7 DEFINITION. Let (P, Q, a) be a complete extension and let b: $Q \to P$ be the right adjoint of a. If the map b is dually residuated, (P, Q, a) will be called a *continuous extension*.

REMARK. Since the map b: $Q \to P$ is both residuated and dually residuated, it has both a left adjoint and a right adjoint. The left adjoint of b is the original map a: $P \to Q$, which is dually residuated. The right adjoint of b is a map c: $P \to Q$ which is residuated. In general, the maps a and c need not be the same. In order to help the reader to keep his bearings, we attempt throughout this article to use a, b and c in the roles indicated. Since a is one-one, b is onto and c is one-one. Thus for each $x \in P$ and $y \in Q$ we have $b(a(x)) = x$, $a(b(y)) \geqslant y$, $b(c(x)) = x$, $c(b(y)) \leqslant y$.

EXAMPLE. Let L and L_1 be complete lattices and let b: $L \to L_1$ be a complete homomorphism from L onto L_1. This implies that b preserves arbitrary joins and meets. Thus, by Remark 1.6, b is residuated and dually residuated. If a: $L_1 \to L$ is the left adjoint of b, then (L_1, L, a) is a continuous extension.

The following proposition justifies the definition of a continuous extension.

2.8 PROPOSITION. *If M is a system of subsets of a poset P, then P is M-continuous if and only if $(P, I_M(P), a_M)$ is a continuous extension. Moreover, if P is M-continuous and if c_M is the right adjoint of b_M (which is the right adjoint of a_M), then for every $x \in P$, $c_M(x) = \{y \mid y \ll_M x\}$.*

PROOF. Assume that P is M-continuous. By Definition 2.3, P is M-complete. Thus, it follows from Lemma 2.6 that $(P, I_M(P), a_M)$ is a complete extension and b_M, the right adjoint of a_M, satisfies the condition that $b_M(\downarrow S) = \bigvee S$ for every $S \in M$. Define the map c_1: $P \to I_M(P)$ by $c_1(x) = \{y \mid y \ll_M x\}$ for each $x \in P$. It is easy to check, using Lemma 1.9, that b_M is dually residuated. Thus we obtain that $(P, I_M(P), a_M)$ is a continuous extension and that $c_M(x) = c_1(x) = \{y \mid y \ll_M x\}$ for each x $\in P$.

Assume that $(P, I_M(P), a_M)$ is a continuous extension. Thus $(P, I_M(P), a_M)$ is a complete extension. Hence, by Lemma 2.6, P is M-complete. Let b_M; $I_M(P) \to P$ be the right adjoint of a_M and let c_M: $P \to I_M(P)$ be the right adjoint of b_M. By Lemma 1.9, for each $x \in P$ and each $S \in M$, $x \leqslant b_M(\downarrow S)$ if and only if $c_M(x) \subseteq \downarrow S$. By Lemma 2.6, $b_M(\downarrow S) = \bigvee S$; thus, for each $x \in P$ and each $S \in M$, $x \leqslant \bigvee S$ if and only if $c_M(x) \subseteq \downarrow S$. This shows that $c_M(x) \subseteq \{y \mid y \ll_M x\}$. Let $y \ll_M x$. Since $x = b_M(c_M(x)) = \bigvee c_M(x)$ and since $c_M(x) \in I_M(P)$, it follows that $y \in c_M(x)$. We conclude that $c_M(x) = \{y \mid y \ll_M x\}$. Since $c_M(x) \in I_M(P)$, it follows that $\{y \mid y \ll_M x\} \in I_M(P)$. Since for each $x \in P$, $b_M(c_M(x)) = x$, we conclude that $\bigvee\{y \mid y \ll_M x\} = x$.

REMARK. If P is M-continuous, b_M and c_M will always play the roles indicated in Proposition 2.8.

2.9 DEFINITION. For each complete extension (P, Q, a) we define a corresponding binary relation on the set P called the *well below relation modulo Q and a* and denoted by $\ll_{Q,a}$. For x, y in P we have that $x \ll_{Q,a} y$ if and only if for each $u \in Q$ the following condition holds: If $y \leqslant b(u)$ then $a(x) \leqslant u$. (Whenever possible we will abbreviate this notation to $\ll_a$.)

Thus the concept of the well below relation can be defined merely under the assumption that a is dually residuated. The following lemma justifies this definition.

2.10 LEMMA. *If P is an M-complete poset and $x, y \in P$, then $x \ll_M y$ if and only if $x \ll_{a_M} y$.*

PROOF. Assume that $x \ll_M y$. Let $S \in M$ satisfy $y \leqslant b_M(\downarrow S)$. By Lemma 2.6 this implies that $y \leqslant \bigvee S$. It follows from Definition 2.2 that $x \leqslant s$ for some $s \in S$. Hence $a(x) \subseteq \downarrow S$, and thus $x \ll_{a_M} y$. The converse is proved in a similar way.

2.11 LEMMA. *Some of the basic properties of a well below relation are*:

(1) *If $x \ll_a y$, then $x \leqslant y$.*

(2) *If $x \ll_a y$ and $y \leqslant z$, then $x \ll_a z$.*

(3) *If $x \ll_a y$ and $z \leqslant x$, then $z \ll_a y$.*

(4) *If $x \ll_a y$ and $y \ll_a z$, then $x \ll_a z$.*

2.12 LEMMA. *If (P, Q, a) is a continuous extension, then for $x, y \in P$, $x \ll_a y$ if and only if $a(x) \leqslant c(y)$.*

PROOF. (1) Assume that $x \ll_a y$. Since $y \leqslant b(c(y))$ it follows from Definition 2.9 that $a(x) \leqslant c(y)$.

(2) The converse is proved in a similar way.

We introduce the following notation for later use.

2.13 DEFINITION. Let (P, Q, a) be a complete extension. For $x \in P$ define $\Downarrow x = \{z \in P \mid z \ll_a x\}$. For $S \subseteq P$ define $\Downarrow S = \bigcup\{\Downarrow x \mid x \in S\}$. (Clearly both $\Downarrow x$ and $\Downarrow S$ belong to $L(P)$.)

In the following theorem we summarise the connection between continuous extensions and dually residuated closure operators. We leave the proof as an exercise.

2.14 THEOREM. *Up to isomorphism, there is a one-one correspondence between continuous extensions and dually residuated closure structures.*

With the continuous extension (P, Q, a) this correspondence associates the closure structure $(Q, a \circ b)$, where b is the right adjoint of a and $c \circ b$ is the right adjoint of $a \circ b$. With the closure structure (Q, g), this correspondence associates the continuous extension $(g(Q), Q, i_g)$. Under this correspondence, isomorphic continuous extensions are carried into isomorphic closure structures and isomorphic closure structures are carried into isomorphic continuous extensions

REMARK. It is easy to check that if $f\colon Q \to Q$ is a dually residuated closure operator and if g is the right adjoint of f, then $g\colon Q^* \to Q^*$ is also a dually residuated closure operator. If we translate this fact to the context of continuous extensions, we obtain the following corollary.

2.15 COROLLARY. *If (P, Q, a) is a continuous extension, then $(c(P)^*, Q^*, i_c)$ is a continuous extension, and $j_{c \circ b}$ is the right adjoint of i_c.*

Now we show that the concept of continuity is equivalent, under certain conditions, to the concept of distributivity. The following results are closely related and overlap with some of the results in [**1**, **6**].

The following definition was introduced by G. Bruns in [**4**].

2.16 DEFINITION. Let M be a system of subsets of a complete lattice P. P is called *M-distributive* if and only if for every family $\{S_i \mid i \in I\} \subseteq M$, $\bigwedge\{\bigvee S_i \mid i \in I\} = \bigvee\{\bigwedge f(I) \mid f \in \prod S_i\}$. (Here $\prod S_i$ denotes the cartesian product of the family $\{S_i \mid i \in I\}$.)

REMARK. If P is M-complete and $I_M(P)$ is a complete lattice, then the meet operation in $I_M(P)$ is necessarily set intersection.

REMARK. If (P, Q, a) is a complete extension and Q is a complete lattice, then P is necessarily a complete lattice. This follows from the fact that b maps Q onto P and b preserves arbitrary joins. In particular, if P is M-complete and $I_M(P)$ is a complete lattice, then P must also be a complete lattice.

2.17 PROPOSITION. *If M is a system of subsets of a complete lattice P such that $I_M(P)$ is a complete lattice, then the following statements are equivalent.*

(1) *P is M-continuous.*

(2) *For each $x \in P$, $x = \bigvee\{y \mid y \ll_M x\}$.*

(3) *P is M-distributive.*

(*By the previous remark it is sufficient to assume that P is M-complete and that $I_M(P)$ is a complete lattice.*)

PROOF. (1) $\Rightarrow$ (2) This follows from Definition 2.3.

(2) $\Rightarrow$ (3) Let $\{S_i \mid i \in I\} \subseteq M$. Since the inequality $\bigvee\{\bigwedge f(I) \mid f \in \prod S_i\} \leqslant \bigwedge\{\bigvee S_i \mid i \in I\}$ holds in any complete lattice, we need only to show the reverse inequality. Let $y \in P$ satisfy $y \ll_M \bigwedge\{\bigvee S_i \mid i \in I\}$. Then for each $i \in I$, $y \ll_M \bigvee S_i$. Hence for each $i \in I$ there exists $s \in S_i$ such that $y \leqslant s$. Now, by the axiom of choice, there exists $f \in \prod S_i$ such that $y \leqslant \bigwedge f(I)$. It follows that $y \leqslant \bigvee\{\bigwedge f(I) \mid f \in \prod S_i\}$. Since y was arbitrary, it follows by (2) that

$$\bigwedge\{\bigvee S_i \mid i \in I\} \leqslant \bigvee\left\{\bigwedge f(I) \mid f \in \prod S_i\right\}.$$

(3) $\Rightarrow$ (1) Let $\{S_i \mid i \in I\} \subseteq M$. First note that

$$\bigcap\{\downarrow S_i \mid i \in I\} = \downarrow\{\bigwedge f(I) \mid f \in \prod S_i\}.$$

If P is M-distributive, then

$$\begin{aligned} b_M\left(\bigcap \{\downarrow S_i \mid i \in I\}\right) &= \bigvee \left(\bigcap \{\downarrow S_i \mid i \in I\}\right) \\ &= \bigvee \left(\downarrow\left\{\bigwedge f(I) \mid f \in \prod S_i\right\}\right) \\ &= \bigvee \left\{\bigwedge f(I) \mid f \in \prod S_i\right\} = \bigwedge\left\{\bigvee S_i \mid i \in I\right\} \\ &= \bigwedge\left\{\bigvee(\downarrow S_i) \mid i \in I\right\} = \bigwedge\left\{b_M(\downarrow S_i) \mid i \in I\right\}. \end{aligned}$$

This shows that b_M preserves arbitrary meets. By Remark 1.6 it now follows that b_M is dually residuated, and thus P is M-continuous.

REMARK. for the special case $M = \mathcal{P}$ this result was expressed by Raney in **[14]**. For $M = \mathcal{D}$, it was found by Day in **[5]**.

2.18 COROLLARY. *Let M be a system of lower subsets of a complete lattice L which is closed under arbitrary intersection. Then P is M-continuous if and only if P is M-distributive.*

3. Construction of continuous extensions. In this section we define the general notion of a well below relation and show how to construct, from a given well below relation, a continuous extension which generates that well below relation. This section is closely related to the section about auxiliary relations in [7].

The proof of the following lemma is elementary and we leave it to the reader.

3.1 LEMMA. *Let P be a set and let $S \subseteq P \times P$ be a reflexive relation. Then the following conditions are equivalent.*

(1) *S is extendible to a partial order on P.*

(2) *The transitive closure of S is a partial order.*

(3) *The (directed) graph of S has no loop with more that one element.*

3.2 DEFINITION. A bounded poset is a poset which has a greatest element, 1, and a least element, 0. Whenever we have a family of bounded posets P_i, where i varies over an indexing set, we will use 1_i and 0_i respectively to denote the greatest and the least elements of the poset P_i.

Let (P, Q, a) be a continuous extension. For each $i \in P$, let $P_i = \{x \in Q \mid b(x) = i\}$. For each $i \in P$, P_i is a bounded poset, the greatest and the least elements of P_i being given by $1_i = a(i)$ and $0_i = c(i)$, respectively. It is obvious that $Q = \bigcup P_i$ and that the sets P_i are pairwise disjoint. Let $x, y \in Q$ satisfy $x \in P_i$ and $y \in P_j$. If $x \leqslant y$ in Q, then $i = b(x) \leqslant b(y) = j$. Conversely, if $i \leqslant j$ in P, then $0_i = c(i) \leqslant c(j) = 0_j$ and $1_i = a(i) \leqslant a(j) = 1_j$. This observation leads us to the following definition.

3.3 DEFINITION. Let P be a poset and for each $i \in P$ let P_i be a bounded poset. Let $Q = \dot{\bigcup} P_i$ be the disjoint union of the sets P_i. A binary relation S on Q is defined to be a *subcontinuous relation* if it satisfies the following three constraints.

(1) If $x, y \in P_i$ for some $i \in P$, then $x\,S\,y$ if and only if $x \leqslant y$ in P_i.

(2) If $x \in P_i$, $y \in P_j$, and $x\,S\,y$, then $i \leqslant j$.

(3) If $i \leqslant j$ in P, then $1_i\,S\,1_j$ and $0_i\,S\,0_j$.

3.4 LEMMA. *Let the poset P, the various posets P_i for $i \in P$, and the set Q satisfy the conditions of Definition 3.3. If S is a subcontinuous relation on Q and R is the transitive closure of S, then R is a partial order on Q.*

PROOF. Let $x_{i_1} S x_{i_2} S \cdots S x_{i_n} = x_{i_1}$ be a loop in S. From condition (2) it follows that $i_1 \leqslant i_2 \leqslant \cdots \leqslant i_n = i_1$. Hence condition (3) of Lemma 3.1 is satisfied.

3.5 DEFINITION. Let the poset P, the various posets P_i for $i \in P$, and the set Q satisfy the conditions of Definition 3.3. If S is a subcontinuous relation on Q and R is the transitive closure of S, then we say that the poset (Q, R) is a *continuous sum* of the posets P_i relative to P. (Since, by Lemma 3.4, R is a partial order, we will sometimes use "$\leqslant$" in place of "R".)

3.6 DEFINITION. Let the poset P, the various posets P_i for $i \in P$, and the set Q satisfy the conditions of Definition 3.3. The *lexicographic sum of the posets P_i relative to the poset P* (see [2]) is a poset (Q, R_1) where R_1 is given by the following rules.

(L_1) If $x, y \in P_i$ for some $i \in P$, then $x R_1 y$ if and only if $x \leqslant y$ in P_i.

(L_2) If $x \in P_i$ and $y \in P_j$ where $i \neq j$, then $x R_1 y$ if and only if $i < j$.

We define the *trivial sum of the posets P_i relative to the poset P* to be the poset (Q, R_0), where R_0 is given by the following rules.

(T_1) If $x, y \in P_i$ for some $i \in P$, then $x R_0 y$ if and only if $x \leqslant y$ in P_i.

(T_2) If $x \in P_i$ and $y \in P_j$ where $i \neq j$, then $x R_0 y$ if and only if $i < j$ and either $x = 0_i$ or $y = 1_j$.

Our next proposition characterizes the various continuous sums of a family of bounded posets P_i relative to a given poset P. Its proof is left to the reader.

3.7 PROPOSITION. *Let the poset P, the various posets P_i for $i \in P$, and the set Q satisfy the conditions of Definition* 3.3. *Lt (Q, R_1), and (Q, R_0) be the lexicographic sum and the trivial sum of the posets P_i relative to the poset P. Let R be a partial order on Q. Then the following two conditions are equivalent.*

(1) *(Q, R) is a continuous sum of the posets P_i relative to P.*

(2) $R_0 \subseteq R \subseteq R_1$.

3.8 PROPOSITION. *Let P be a poset and let $\{P_i \mid i \in I\}$ be a family of bounded posets indexed by P. Let Q be a continuous sum of the posets P_i relative to P. Let the map a: $P \to Q$ be given by $a(i) = 1_i$ for each $i \in P$. Then (P, Q, a) is a continuous extension. The right adjoint of a is the map b: $Q \to P$ given by $b(x) = i$ if $x \in P_i$. The right adjoint of b is the map c: $P \to Q$ given by $c(i) = 0_i$ for each $i \in P$.*

PROOF. Let $x \in Q$ and $i \in P$ satisfy $x \in P_i$. By the definitions of a and b, $a(b(x)) = a(i) = 1_i$. since $x \leqslant 1_i$ in P_i, it follows from Definition 3.3(1) that $1_i \geqslant x$ in Q and thus $a(b(x)) \geqslant x$. Conversely, $b(a(i)) = b(1_i) = i$. Thus we have shown that (a, b) is a Galois connection from P to Q. In the same way it can be shown that (b, c) is a Galois connection from Q to P.

3.9 DEFINITION. Let $(P, \leqslant)$ be a poset and let S be a binary relation on P. We say that S is a *subrelation of* $\leqslant$ if, for all $x, y \in P$, $x S y$ implies $x \leqslant y$.

3.10 DEFINITION. Let $(P, \leqslant)$ be a poset and let R be a binary relation on the set P. We say that R is a *well below relation* on P if, for all $x, y, z, w \in P$, the following two conditions are satisfied. (See Definition 1.9 in [7].)

(1) $x R y$ implies $x \leqslant y$.

(2) $x \leqslant y R z \leqslant w$ implies $x R w$.

Now we will show that, given a poset $(P, \leqslant)$ and a well below relation on P, we can, in a natural way, construct a continuous extension (P, Q, a) such that the well below relation which corresponds to this continuous extension is the given well below relation.

3.11 PROPOSITION. *Let* $(P, \leqslant)$ *be a poset and let* R *be a binary relation on* P. *The relation* R *is a well below relation on* P *if and only if there exists a continuous extension* (P, Q, a) *such that* R *coincides with the relation* $\ll_a$.

PROOF. If (P, Q, a) is a continuous extension, then $\ll_a$ is a well below relation on P by Lemma 2.11. Conversely, assume that R is a well below relation on P. For each $i \in P$ let P_i be the following poset: If $i\,R\,i$, the P_i is a one-element poset. If not $i\,R\,i$, then P_i is a two-element chain. (When $i\,R\,i$, it will be convenient to permit the unique element of P_i to be denoted both by 0_i and by 1_i.) Let $Q = \dot{\bigcup}\, P_i$ and let S be the least relation on Q that satisfies the following three conditions.

(1) $x\,S\,x$ for each $x \in Q$, and for each $i \in P$, $0_i\,S\,1_i$.

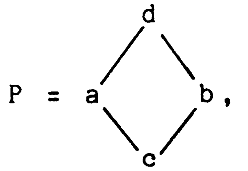
(2) If $i \leqslant j$ in P, then $0_i\,S\,0_j$ and $1_i\,S\,1_j$.

(3) If $i\,R\,j$, then $1_i\,S\,0_j$.

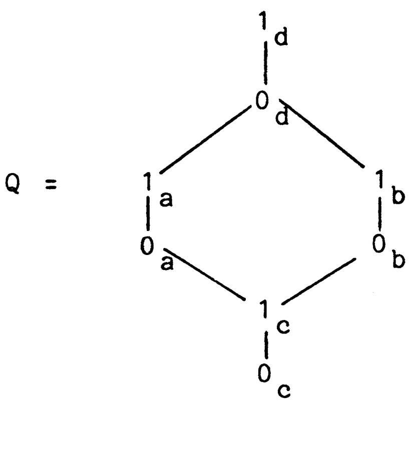
It follows from Definition 3.3 that S is a subcontinuous relation on Q. Let Q be ordered by the transitive closure of S. Then, by Proposition 3.8, (P, Q, a) is a continuous extension where $a(i) = 1_i$ for each $i \in P$.

To complete the proof we will show that indeed $R = \ll_a$. Let $i, j \in P$. By Lemma 2.12, $i \ll_a j$ if and only if $a(i) \leqslant c(j)$. By Proposition 3.8, $a(i) = 1_i$ and $c(i) = 0_i$. Thus we obtain that $i \ll_a j$ if and only if $1_i \leqslant 0_j$. If $i\,R\,j$ then $1_i\,S\,0_j$ and thus $1_i \leqslant 0_j$. Conversely, assume that $1_i \leqslant 0_j$. Then there exist $x_0, x_1, \ldots, x_n \in P$ (where $x_k = 0_i$ or $x_k = 1_i$ for each k) such that $1_i = x_0\,S\,x_1\,S \cdots S\,x_n = 0_j$. Let $1 \leqslant k < n$ be such that $x_k = 1_i$ and $x_{k+1} = 0_i$. Thus $i \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_k\,S\,i_{k+1} \leqslant \cdots \leqslant j$. It follows that $i\,R\,j$.

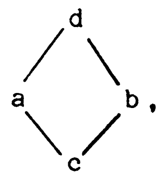
3.12 EXAMPLE. (1) Let n be a natural number. Let $(P, \leqslant)$ be a poset and define the following relation $\ll_n$ on P: For every $x, y \in P$, $x \ll_n y$ if and only if there exists a chain of n elements between x and y. For example, for $n = 1$: If



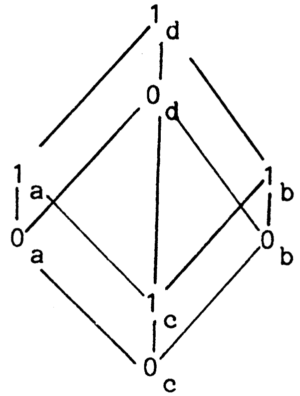
then



For $n = 2$: If

$P =$ 

then

$Q =$ 

This example shows that Q is not necessarily a complete lattice even if P is complete.

(2) Let $(P, \leqslant)$ be a poset and let $A \subseteq P$. We define a subrelation R of $\leqslant$ by: $a\, R\, a$ for each $a \in A$. Let the well below relation on P generated by R be denoted by $\ll_A$. Thus $x \ll_A y$ if and only if there exists $a \in A$ such that $x \leqslant a \leqslant y$.

This example shows that for any subset A of P there is a well below relation having the property that the compact elements relative to that well below relation (that is, elements are well below themselves) are precisely the elements of the set A.

3.13 Definition. Let P and Q be two posets such that $P \subseteq Q$ and such that the order relation on P is equal to the order relation on Q restricted to P. If for each $x \in Q$, $x = \bigvee(\downarrow x \cap P)$, then we say that P is *join dense* in Q.

3.14 Definition. An extension (P, Q, a) is called *join dense* if $a(P)$ is join dense in Q.

Join dense extensions were studied by J. Schmidt in [**16, 17**]. The following theorem characterizes the well below relations which correspond to M-continuous extensions.

3.15 Proposition. *Let $(P, \leqslant)$ be a poset and let $\ll$ be a well below relation on P. Then the following conditions are equivalent.*

(1) *For each $x \in P$, $x = \bigvee\{y \mid y \ll x\}$. (In the terminology of* [7] *this means that $\ll$ is approximating.)*

(2) *There exists a system of subsets M of P such that P is M-continuous and such that for all $u, v \in P$, $u \ll v$ if and only if $u \ll_M v$.*

(3) *There exist a poset Q and an order imbedding $a\colon P \to Q$ such that (P, Q, a) is a continuous extension, $a(p)$ is join dense in Q, and the relation $\ll$ coincides with the relation $\ll_a$.*

PROOF. (1) $\Rightarrow$ (2) Assume that for every $x \in P$, $x = \bigvee\{y \mid y \ll x\}$. Define the family M of subsets of P as follows: $S \in M$ if and only if $S = \downarrow x$ or $S = \{y \mid y \ll x\}$ for some $x \in P$. It is easy to show that for $u, v \in P$, $u \ll v$ holds if and only if $u \ll_M v$. By Definition 2.3 it follows that P is indeed M-continuous.

(2) $\Rightarrow$ (3) This follows from Proposition 2.8, where $Q = I_M(P)$ and $a = a_M$. Obviously $a_M(P)$ is join dense in Q.

(3) $\Rightarrow$ (1) Let $x, z \in P$ and let z be an upper bound for the set $\{y \mid y \ll_a x\}$. By Lemma 2.12, $y \ll_a x$ holds if and only if $a(y) \leqslant c(x)$. Since $a(P)$ is join dense in Q it follows that $c(x) = \bigvee\{a(y) \mid a(y) \leqslant c(x)\}$. Since $y \leqslant z$ holds if and only if $a(y) \leqslant a(z)$, it now follows that $c(x) \leqslant a(z)$. Thus $x = b(c(x)) \leqslant b(a(z)) = z$. It follows that $x \leqslant z$ and thus x is the least upper bound of the set $\{y \mid y \ll_a x\}$.

4. Continuity and complete distributivity. In this section we introduce the concept of a strongly continuous extension and show a connection between strong continuity and complete distributivity. It is shown that in the special cases where $M = \mathcal{F}$, $\mathcal{D}$ or $\mathcal{P}$, continuity coincides with strong continuity.

4.1 DEFINITION. To each complete extension (P, Q, a) there corresponds a family J_a $(= J_{Q,a})$ of lower subsets of P given by $J_a = \{A \subseteq P \mid A = \downarrow A$ and for each $u \in Q$, if $a^{-1}(\downarrow u) \subseteq A$, then $b(u) \in A\}$. (We will omit the subscript, once a is fixed throughout a discussion.)

4.2 DEFINITION. Let (P, Q, a) be a complete extension. For each subset $S \subseteq P$ we define $\bar{S}\,(= \bar{S}^{Q,a}) = \bigcap\{A \subseteq P \mid A \in J$ and $S \subseteq A\}$.

REMARK. The operation $S \to \bar{S}$ is a *closure operation* on P. Moreover, a set has the form $\bar{S}$ for some $S \subseteq P$ if and only if it belongs to the family J. The sets in J are called the *closed sets* of the closure operator $S \to \bar{S}$.

In case the poset P is M-complete, the family J_{a_M} will be denoted by J_M. Any set in the family J_M will be called an M-ideal (see [**16**]). Thus a subset $A \subseteq P$ is an M-ideal if and only if $A = \downarrow A$ and A is M-join closed (this means that for each $S \in M$, $S \subseteq A$ implies that $\bigvee S \in A$). For $S \subseteq P$ we will denote the set $\bar{S}^{a_M}$ by $\bar{S}^M$.

REMARK. The $\mathcal{D}$-ideals coincide with the "Scott closed" sets. (See [**10, 18**].)

The $\mathcal{F}$-ideals are just the ideals in the usual lattice theoretic sense.

The $\mathcal{P}$-ideals are just the principal ideals.

4.3 LEMMA. *If (P, Q, a) is a continuous extension, then the following conditions are equivalent.*

(1) $A \in J$.

(2) $A = \downarrow A$ *and for each* $x \in P$, *if* $\{y \mid y \ll_a x\} \subseteq A$, *the* $x \in A$.

PROOF (1) $\Rightarrow$ (2) By Lemma 2.12, $\{y \mid y \ll_a x\} = a^{-1}(\downarrow c(x))$. Thus, if $\{y \mid y \ll_a x\} \subseteq A$, it follows from Definition 4.1 that $x = b(c(x)) \in A$.

(2) $\Rightarrow$ (1) Let $u \in Q$ be such that $a^{-1}(\downarrow u) \subseteq A$. Since $c(b(u)) \leqslant u$, it follows that $a^{-1}(\downarrow(c(b(u)))) \subseteq \downarrow A = A$. Let $x = b(u)$. It follows from Lemma 2.12 that $a^{-1}(\downarrow c(b(u))) = \{y \mid y \ll_a x\}$. Thus by (2) we obtain that $b(u) = x \in A$.

4.4 COROLLARY. *If (P, Q, a) is a continuous extension, then the following conditions are equivalent.*

(1) $P \setminus B \in J$.

(2) $B = \uparrow B$ *and for each* $x \in B$ *there exists* $y \in B$ *such that* $y \ll_a x$.

4.5 DEFINITION. To each complete extension (P, Q, a) there corresponds a family Γ $(= \Gamma^{Q,a})$ of lower subsets of P given by $\Gamma = \{A \subseteq P \mid A = \downarrow A$ and if $x \in A$, then there exists $y \in A$ such that $x \ll_a y\}$.

REMARK. (1) $A \in \Gamma$ if and only if $A = \twoheaddownarrow A$.

(2) The family Γ is closed under arbitrary union.

4.6 DEFINITION. Let (P, Q, a) be the complete extension. For each subset $S \subseteq P$ we define

$$\mathring{S} \, (= \mathring{S}^{Q,a}) = \bigcup \, \{A \in \Gamma \mid A \subseteq S\}.$$

REMARK. The operation $S \mapsto \mathring{S}$ is an anti-closure operation on P. Moreover, a set has the form $\mathring{S}$ for some $S \subseteq P$ if and only if it belongs to the family Γ. The sets in Γ are called the *open sets* of the anti-closure operator $S \mapsto S$.

4.7 PROPOSITION. *Let P be an M-complete poset. If for every $x \in P$, the set $\{T \in I_M(P) \mid x \in \overline{T}\}$ has a least element, then P is M-continuous.*

PROOF. Define the map $c\colon P \to I_M(P)$ by $c(x) = \min\{T \in I_M(P) \mid x \in \overline{T}\}$ for each $x \in P$. It is easy to show that c is the right adjoint of b and thus, by Lemma 1.9, b is dually residuated. This will show, by Definition 2.3, that P is M-continuous.

The following theorem by J. Lawson (Theorem 5.6 in [**10**]) guided the thoughts expressed in this article.

4.8 THEOREM. *If P is $\mathcal{D}$-complete, then P is $\mathcal{D}$-continuous if and only if $J_{\mathcal{D}}(P)$ is $\mathcal{P}$-continuous.*

The analog of Lawson's theorem obtained when $\mathcal{D}$ is replaced by $\mathcal{F}$ is

4.9 THEOREM. *If P is $\mathcal{F}$-complete, then P is $\mathcal{F}$-continuous if and only if $J_{\mathcal{F}}(P)$ is $\mathcal{P}$-continuous.* (*This result follows from Theorem* 6.5 *and is closely related to a result by J. Martinez* [**12**].)

4.10 DEFINITION. Let (P, Q, a) be a continuous extension. We define the functions

$$K_a \, (= K_{Q,a})\colon L(P) \to L(P) \qquad \text{and} \qquad R_a \, (= R_{Q,a})\colon L(P) \to L(P)$$

by

$$K_a(S) = \{b(x) \mid a^{-1}(\downarrow x) \subseteq S\}$$

and

$$R_a(S) = \twoheaddownarrow S = \{x \mid x \ll_a s \text{ for some } s \in S\}.$$

(We will omit the subscript a, once a is fixed throughout a discussion.)

The basic properties of R and K are expressed in the following lemma.

4.11 LEMMA. *For each $S, S_1, S_2 \in L(P)$ we have*

(1) $K(S) = \{x \mid a^{-1}(\downarrow c(x)) \subseteq S\}$ *and* $R(S) = a^{-1}(\downarrow c(S))$.

(2) $K(S) \in L(P)$ *and* $R(S) \in L(P)$.

(3) *If* $S_1 \subseteq S_2$, *then* $K(S_1) \subseteq K(S_2)$ *and* $R(S_1) \subseteq R(S_2)$.

(4) $S \subseteq K(S) \subseteq \overline{S}$ *and* $\mathring{S} \subseteq R(S) \subseteq S$.

(5) $S \subseteq K(R(S))$ *and* $R(K(S)) \subseteq S$.

(6) (K, R) *is a Galois connection from* $L(P)$ *to* $L(P)$. *Thus* K *preserves arbitrary intersections and* R *preserves arbitrary unions.*

(7) $S_1 \subseteq K(S_2)$ *if and only if* $R(S_1) \subseteq S_2$. *Thus for each* $y \in P$, $y \in K(S)$ *if and only if* $R(\downarrow y) \subseteq S$.

(8) $S = \bar{S}$ *if and only if* $K(S) = S$.

(9) $S = \mathring{S}$ *if and only if* $R(S) = S$.

(10) $K(K(S)) = K(S)$ *if and only if* $K(S) = \bar{S}$.

(11) $R(R(S)) = R(S)$ *if and only if* $R(S) = \mathring{S}$.

PROOF. (1) We will show that $K(S) = \{x \mid a^{-1}(\downarrow c(x)) \subseteq S\}$. Assume first that $x \in K(S)$. Thus $x = b(y)$ for some y such that $a^{-1}(\downarrow y) \subseteq S$. We need to show that $a^{-1}(\downarrow c(x)) \subseteq S$; let $z \in a^{-1}(\downarrow c(x))$. Then $a(z) \leqslant c(x) = c(b(y)) \leqslant y$. Thus $z \in a^{-1}(\downarrow y) \subseteq S$. This proves one inclusion.

Next assume that $a^{-1}(\downarrow c(x)) \subseteq S$ and let $y = c(x)$. Then $a^{-1}(\downarrow y) \subseteq S$ and $x = b(c(x)) = b(y)$. This shows that $x \in K(S)$.

The proof that $R(S) = a^{-1}(\downarrow c(S))$ is accomplished by a direct use of Lemma 2.12.

(2) This is based on (1) and is left to the reader.

(3) and (4). These follow from the definitions and are left as an exercise.

(5) First we show that $S \subseteq K(R(S))$. Let $x \in S$. Then $a^{-1}(\downarrow c(x)) = R(S)$ by (1) and since $x = b(c(x))$, it follows that $x \in K(R(S))$.

Next we prove that $R(K(S)) \subseteq S$. Let $y \in R(K(S))$; then $y \ll_a u$ for some $u \in K(S)$. By definition of K this means that there exists $v \in S$ such that $y \ll_a b(v) = u$ and $a^{-1}(\downarrow v) \subseteq S$. If we apply Definition 2.9, we obtain that $a(y) \leqslant v$ and thus $y \in S$.

(6) and (7). These follow from (5).

(8) Assume first that $S = \bar{S}$. Then $S \in J$. By (4), $S \subseteq K(S)$; thus we need only show that $K(S) \subseteq S$. Let $y \in K(S)$. This means that $y = b(x)$ for some x such that $a^{-1}(\downarrow x) \subseteq S$. It follows from Definition 4.1 that $y \in \bar{S} = S$.

Next assume that $S = K(S)$. Thus, if $a^{-1}(\downarrow u) \subseteq S$ for some $u \in Q$, then $b(u) \in K(S) = S$. Also $S = \downarrow S$, since $S \in L(P)$. Thus by Definition 4.1, $S \in J$. Therefore $S = \bar{S}$.

(9) This follows from Definition 4.5.

(10) This follows from (8).

(11) This follows from (9).

4.12 LEMMA. *Let* (g, f) *be a Galois connection from* Q *to* Q *and let* $x \in Q$. *Under the hypothesis that* $f^2(x) \leqslant g(f^2(x)) = g^2(f^2(x))$, *the following conditions are equivalent.*

(1) $f^2(x) \leqslant g(f^2(x))$.

(2) $f(x) \leqslant g^2(f^2(x))$.

(3) $f(x) \leqslant g(f^2(x))$.

(4) $x \leqslant g^2(f^2(x))$.

(5) $x \leqslant g(f^2(x))$.

(6) $f(x) \leqslant f^2(x)$.

PROOF. The proof is based on Lemma 1.9(2).

In a similar way we obtain the following result.

4.13 LEMMA. *Let* (g, f) *be a Galois connection from* Q *to* Q *and let* $x \in Q$. *If* $f^2(g^2(x)) = f(g^2(x)) \leq g^2(x)$, *then* $g^2(x) \leq g(x)$.

Applying these lemmas to the special case of (K, R) we obtain the following corollary.

4.14 COROLLARY. *Let* (P, Q, a) *be a continuous extension and let* $S \in L(P)$. *If* $K(R^2(S)) = K^2(R^2(S))$, *then* $R^2(S) = R(S)$, *and if* $R(K^2(S)) = R^2(K^2(S))$, *then* $K^2(S) = K(S)$.

PROOF. The first part of the proof follows from Lemma 4.12 and Lemma 4.11(4). The other part is proved in a similar way, using Lemma 4.13 instead of Lemma 4.12.

REMARK. If P is M-continuous, then K_{a_M} and R_{a_M} will be denoted by K_M and R_M. Even the subscript M will be omitted once M is fixed throughout a discussion. Thus for $S \in L(P)$ we have $K(S) = \{\bigvee T \mid T \in M \text{ and } T \subseteq S\}$.

4.15 DEFINITION. A continuous extension (P, Q, a) is defined to have the *one-step closure property* if, for every $S \in L(P)$, $K(S) = \bar{S}$.

4.16 DEFINITION. A continuous extension (P, Q, a) is defined to have the *one-step anti-closure property* if, for every $S \in L(P)$, $R(S) = \mathring{S}$.

REMARK. By Lemma 4.11(8), (9) (P, Q, a) has the one-step (anti-)closure property if and only if $K(S) \in J$ $(R(S) \in \Gamma)$ for every $S \in L(P)$.

4.17 DEFINITION. A complete extension (P, Q, a) is defined to have the *interpolation property* if $(\ll_a)^2 = (\ll_a)$. This implies that for every $x, y \in P$ such that $x \ll_a y$, there exists $z \in P$ such that $x \ll_a z \ll_a y$.

REMARK. Clearly (P, Q, a) has the interpolation property if and only if $R^2 = R$.

4.18 THEOREM. *If* (P, Q, a) *is a continuous extension, then the following conditions are equivalent.*

(1) *For each* $y \in P$, $R^2(\downarrow y) = R(\downarrow y)$.

(2) *For each* $y \in P$, $R(\downarrow y) = (\downarrow^\circ y)$.

(3) $R^2 = R$. *That is*, (P, Q, a) *has the interpolation property*.

(4) (P, Q, a) *has the one-step anti-closure property*.

(5) $K^2 = K$.

(6) (P, Q, a) *has the one-step closure property*.

(7) *For each* $y \in P$, $K(R^2(\downarrow y)) = \underline{K^2(R^2(\downarrow y))}$.

(8) *For each* $y \in P$, $K(R^2(\downarrow y)) = R^2(\downarrow y)$.

(9) $(K(L(P)), L(P), i_K)$ *is a complete extension, and* $K(L(P))/K$ *is the right adjoint of* i_K.

(10) $(R(L(P))^*, L(P)^*, i_R)$ *is a complete extension, and* $R(L(P))/R$ *is the right adjoint of* i_R.

PROOF. (1) $\Leftrightarrow$ (2) This follows from Lemma 4.11(11).

(1) $\Rightarrow$ (3) Assume that for every $y \in P$, $R^2(\downarrow y) = R(\downarrow y)$, and let $S \in L(P)$. We will show that $R(S) \subseteq R^2(S)$ and thus complete the proof, since by Lemma 4.11(4),

$R^2(S) \subseteq R(S)$. By Definition 4.10, $R(S) = \bigcup\{R(\downarrow y) \mid y \in S\}$. Thus, by assumption (1), $R(S) = \bigcup\{R^2(\downarrow y) \mid y \in S\}$. Since, by Lemma 4.11(3), $R^2(\downarrow y) \subseteq R^2(S)$ for every $y \in S$, we obtain that $R(S) \subseteq R^2(S)$.

(3) $\Leftrightarrow$ (4) This follows from Lamma 4.11(11).

(3) $\Rightarrow$ (5) This follows from Lemma 1.14 and Lemma 4.11(6).

(5) $\Leftrightarrow$ (6) This follows from Lemma 4.11(10).

(6) $\Rightarrow$ (7) If $K(S) = \overline{S}$ for every $S \in L(P)$, then, in particular, for every $y \in P$,

$$K^2(R^2(\downarrow y)) = \overline{\overline{(R^2(\downarrow y))}} = \overline{R^2(\downarrow y)} = K(R^2(\downarrow y)).$$

(7) $\Leftrightarrow$ (8) This follows from Lemma 4.11(10).

(7) $\Rightarrow$ (1) This follows from Corollary 4.14.

(4) $\Rightarrow$ (9) By (4), K: $L(P) \to L(P)$, is a closure operator. It follows that $(K(L(P)), L(P), i_K)$ is a complete extension and $K(L(P))/K$ is the right adjoint of i_K.

(6) $\Rightarrow$ (10) By (6), R: $L(P) \to L(P)$, is an anti-closure operator. Thus R: $L(P)^* \to L(P)^*$ is a closure operator. The rest of the proof is exactly like the proof that (4) $\Rightarrow$ (9).

(9) $\Rightarrow$ (5) By (9), $K(L(P))/K$ is the right adjoint of i_K. Since $K = i_K \circ K(L(P))/K$, we conclude that K is a closure operator and thus $K^2 = K$.

(10) $\Rightarrow$ (3) This is proved in the same way as (9) $\Rightarrow$ (5).

4.19 DEFINITION. A continuous extension, which satisfies any one of the equivalent conditions of Theorem 4.18, is called a *strongly continuous extension*.

4.20 DEFINITION. An M-continuous poset is called *strongly M-continuous* if $(P, I_M(P), a_M)$ is a strongly continuous extension.

4.21 LEMMA. *Let M be either $\mathfrak{D}$ or $\mathfrak{F}$ or $\mathfrak{P}$. If P is M-continuous, then P is strongly M-continuous.*

PROOF. We have to show that the interpolation property holds in each case and thus, by Definition 4.19, strong continuity will be established.

It is well known that the interpolation property holds in any $\mathfrak{D}$-continuous poset (see [**10**]). G. Raney proved in [**14**] that the interpolation property holds (for the corresponding well below relation) in any $\mathfrak{P}$-continuous poset. The proof that the interpolation property holds in any $\mathfrak{F}$-continuous poset is standard and is left as an exercise.

4.22 COROLLARY. *Let P be a $\mathfrak{D}$-continuous poset and $S \subseteq P$. Then $\overline{S}$, the "Scott closure" of S in P, satisfies $\overline{S} = \{\bigvee D \mid D \subseteq \downarrow S$, D directed$\}$.*

Before going further let us examine some examples.

4.23 EXAMPLE. Example 3.12 ($n = 1$) is a simple example of a continuous extension (P, Q, a) which is not strongly continuous. This can be easily verified by noticing that the interpolation property does not hold.

4.24 EXAMPLE. Let P be a complete lattice with the property that whenever x, $y \in P$ satisfy $x < y$, then there exists $z \in P$ such that $x < z < y$. Define $M = \{\downarrow x \mid x \in P\} \cup \{\{y \mid y < x\} \mid x \in P\}$. We will show that for each $y \in P, \bigvee\{x \in P \mid x < y\} = y$. Obviously $\bigvee\{x \mid x < y\} \leqslant y$. If $w = \bigvee\{x \mid x < y\} < y$, then there exists $z \in P$ such that $w < z < y$ and we have a contradiction.

It is also easy to check that for each $x, y \in P$, $x \ll_M y$ if and only if $x < y$. By the definition of M, for each $y \in P$, $\{x \mid x < y\} \in M$. Thus, by Definition 2.3, P is M-continuous. Since, by assumption, the interpolation property is satisfied, we conclude that P is strongly M-continuous.

5. Consequences of strong continuity. Several results in this section were obtained independently by H. Bandelt and M. Erne in [**1**].

5.1 PROPOSITION. *If (P, Q, a) is a strongly continuous extension, then the following conditions hold.*

(1) *$(J, L(P), i_K)$ is a continuous extenson, J/K is the right adjoint of i_K and R/J is the right adjoint of J/K.*

(2) *$(\Gamma^*, L(P)^*, i_R)$ is a continuous extension, Γ^*/R is the right adjoint of i_R, and K/Γ^* is the right adjoint of Γ^*/R.*

(3) *J is order isomorphic to Γ.*

(4) *J is a completely distributive complete lattice.*

(5) *Γ is a completely distributive complete lattice.*

PROOF. (1) By Theorem 4.18(9), $(J, L(P), i_K)$ is a complete extension and J/K is the right adjoint of i_K. Since, by Lemma 4.11(6), K $(= i_K \circ J/K)$, is dually residuated, it follows, from Theorem 2.14, that $(J, L(P), i_K)$ is a continuous extension, and that R/J is the right adjoint of J/K.

(2) The proof of (2) is similar to that of (1) and is left as an exercise.

(3) By Theorem 4.18(4), (6), $\text{Im}(R) = \Gamma$ and $\text{Im}(K) = J$. By Lemma 4.11(6), (K, R) forms a Galois connection. Thus by Corollary 1.13(1), J is order isomorphic to Γ.

(4) Since the meet operation in J is the same as the meet operation in $L(P)$ (i.e. set intersection), and since by Lemma 4.11(6) K is dually residuated, it follows from Remark 1.6 that J/K preserves arbitrary meets. By Theorem 4.18(6), K is a closure operator and thus J/K is the right adjoint of i_K. It follows from Remark 1.6 that J/K preserves arbitrary joins. Since $L(P)$ is a complete ring of sets and thus, in particular, completely distributive, it follows that $J/K(L(P)) = J$ is completely distributive.

(5) This follows from (3) and (4).

5.2 EXAMPLE. Let P be a poset and let H be the system of those subsets of P which have a least upper bound. The following conditions are equivalent.

(1) P is H-continuous.

(2) $J_H(P)$ is completely distributive complete lattice.

(3) P can be imbedded in a completely distributive complete lattice L in such a way that all existing meets and joins are preserved and such that P is join dense in L.

PROOF. (1) $\Rightarrow$ (2) This will follow from Proposition 5.1 if we can prove that the relation $\ll_H$ has the interpolation property. Let $x, y \in P$ satisfy $x \ll_H y$. Since P is H-continuous, $y = \bigvee\{z \mid z \ll_H y\}$ and for each such z, $z = \bigvee\{w \mid w \ll_H z\}$. Let $S = \{w \mid \text{there exists } z \text{ such that } w \ll_H z \ll_H y\}$. It follws that $\bigvee S = y$. Thus, by the definition of H, $S \in H$. Since $x \ll_H y$, it follows that there exist $w, z \in P$ such that $x \leqslant w \ll_H z \ll_H y$. Thus $z \ll_H z \ll_H y$.

(2) $\Rightarrow$ (3) It is easily verified that the imbedding a ($= a_H$): $P \to J_H$ preserves all (existing) meets. Let $S \subseteq P$ and assume that $\bigvee S$ exists. Since $S \subseteq \bigvee_J \{a(s) \mid s \in S\}$ and $S \in H$, it follows that $\bigvee S \in \bigvee_J \{a(s) \mid s \in S\}$. Thus $a(\bigvee S) = \downarrow(\bigvee S) \leqslant \bigvee_J \{a(s) \mid s \in S\}$. Conversely, since $\downarrow(\bigvee S) \in J$ and $a(s) \subseteq \downarrow(\bigvee S)$ for each $s \in S$, it follows that $\bigvee_J \{a(s) \mid s \in S\} \leqslant \downarrow(\bigvee S) = a(\bigvee S)$. Thus we have shown that the map a preserves all existing joins. Moreover, since for each $A \in J$,

$$A = \bigvee_J \{\downarrow x \mid x \in A\},$$

it follows that $a(P)$ is join dense in J.

(3) $\Rightarrow$ (1) From the definition of the subset system H it follows that P is H-complete. Let $x \in P$. Since L is completely distributive it follows that

(1) $$x = \bigvee_L \{y \in L \mid y \ll_{\mathcal{P}} x\}.$$

Since P is join dense in L we obtain that for each $y \in L$

(2) $$y = \bigvee_L \{z \in P \mid z \leqslant y\}.$$

Combining (1) and (2) it follows that

(3) $$x = \bigvee_L \{z \in P \mid z \ll_{\mathcal{P}} x\}.$$

Let $z, x \in P$ such that $z \ll_{\mathcal{P}} x$ (in L). Since the imbedding of P in L preserves all existing joins, it follows that $z \ll_H x$. Thus we obtain that from (3),

(4) $$x = \bigvee_L \{z \in P \mid z \ll_H x\}.$$

Since $x \in P$, it now follows that

(5) $$x = \bigvee_P \{z \in P \mid z \ll_H x\}.$$

By the defintion of H we obtain that $\{z \in P \mid z \ll_H x\} \in H$. Thus, by Definition 2.3, P is H-continuous.

REMARK. The previous example is closely related to the work of S. Jansen (see [**9**]).

Proposition 5.1 helps us to gain some insight into the work done by Bruns in [**4**]. If P is strongly M-continuous, then J ($= J_M(P)$) is a completely distributive complete lattice and P is join-densely imbedded in J (by the map a_M), in such a way that all existing meets and M-joins are preserved. By Raney's result [**14**], every completely distributive complete lattice can be imbedded in a direct product of complete chains in such a way that arbitrary meets and joins are preserved. Thus we arrive at the following result.

5.3 COROLLARY. *If P is a strongly M-continuous poset, then P can be imbedded in a direct product of complete chains in such a way that all existing meets and M-joins are preserved.*

5.4 DEFINITION. Let P be a complete lattice and let M be a system of subsets of P. M is called *distributive-closed* if the following two conditions are satisfied.

(1) If $\{S_i \mid i \in I\} \subseteq M$, then $\{\bigwedge f(I) \mid f \in \Pi S_i\} \in M$.

(2) If $\{S_i \mid i \in I\} \subseteq M$, and $\{\bigvee S_i \mid i \in I\} \in M$, then $\bigcup \{S_i \mid i \in I\} \in M$.

In [4] Bruns shows that if P is a complete lattice and M is a distributive-closed subset system, then the corresponding well below relation has the interpolation property. Using this fact we now obtain the main result in [4].

5.5 COROLLARY. *If M is a distributive-closed subset system of an M-distributive complete lattice P, then P can be imbedded in a direct product of complete chains in such a way that arbitrary meets and M-joins are preserved.*

PROOF. Define the subset system M_1 by $M_1 = \{\downarrow S \mid S \in M\}$. The proof of the following observations is based on the fact that for each $S \subseteq P$, $\bigvee(\downarrow S) = \bigvee S$.

(1) Since P is M-distributive, P is also M_1-distributive.

(2) Since M is a distributive-closed system of subsets of P, it follows that M_1 is closed under arbitrary intersection.

(3) $\ll_M = \ll_{M_1}$ and thus $\ll_{M_1}$ has the interpolation property. Thus P is M_1-distributive, and M_1 is a system of lower subsets of P which is closed under arbitrary intersection. By Corollary 2.18, P is M_1-continuous. Furthermore, by (3) we conclude that P is even strongly M_1-continuous. By Corollary 5.3, P can be imbedded in a direct product of complete chains in such a way that arbitrary meets and M_1-joins (and thus also M-joins) are preserved.

In Proposition 5.1 we have shown that if (P, Q, a) is a strongly continuous extension, then J is a completely distributive complete lattice. In the following proposition we examine some conditions equivalent to this necessary condition. Then, making use of Lawson's Theorem (Theorem 4.8), we show an application of these results in the $\mathfrak{D}$-continuous case.

5.6 PROPOSITION. *If P is M-complete, then the following conditions are equivalent.*

(1) *J is a completely distributive complete lattice.*

(2) *If $\{S_i \mid i \in I\} \subseteq L(P)$, then $\bigcap \{\overline{S_i} \mid i \in I\} = \overline{\bigcap \{S_i \mid i \in I\}}$; that is, the closure operator preserves arbitrary meets.*

(3) *$(J, L(P), i_K)$ is a continuous extension.*

(4) *For every $A \in J$, $\min\{S \in L(P) \mid A \subseteq \overline{S}\}$ exists.*

PROOF. Since $S_i \subseteq \overline{S_i}$ for each $i \in I$, $\bigcap \{S_i \mid i \in I\} \subseteq \bigcap \{\overline{S_i} \mid i \in I\}$. Thus,

$$\overline{\bigcap \{S_i \mid i \in I\}} \subseteq \overline{\bigcap \{\overline{S_i} \mid i \in I\}} = \bigcap \{\overline{S_i} \mid i \in I\}.$$

To prove the other inclusion we use G. Raney's result on completely distributive lattices [14], which, if applied to our case, means that for each $B \in J$, $B = \bigvee_J \{C \in J \mid C \ll_{\mathcal{P}} B\}$. Since $C = \bigvee_J \{\downarrow x \mid x \in C\}$, we conclude that for each $B \in J$, $B = \bigvee_J \{\downarrow x \mid x \ll_{\mathcal{P}} B\}$. Thus to prove that $\bigcap \{\overline{S_i} \mid i \in I\} \subseteq \overline{\bigcap \{S_i \mid i \in I\}}$, it will suffice to show that for each $x \in P$, if $\downarrow x \ll_{\mathcal{P}} \bigcap \{\overline{S_i} \mid i \in I\}$, then

$$\downarrow x \subseteq \bigcap \{S_i \mid i \in I\}.$$

Let $\downarrow x \ll_{\mathcal{S}} \bigcap \{\bar{S}_i \mid i \in I\}$. It follows that for each $i \in I$, $\downarrow x \ll_{\mathcal{S}} \bar{S}_i$. Since $S_i \subseteq \bigvee_J \{\downarrow y \mid y \in S_i\} \subseteq \bar{S}_i$, and since $\bigvee_J \{\downarrow y \mid y \in S_i\} \in J$, we obtain that $\downarrow x \ll_{\mathcal{S}} \bar{S}_i = \bigvee_J \{\downarrow y \mid y \in S_i\}$. By Definition 2.2, we conclude that for each $i \in I$ there exists $y \in S_i$ such that $\downarrow x \subseteq \downarrow y \subseteq S_i$. It follows that $\downarrow x \subseteq \bigcap \{S_i \mid i \in I\}$. Thus (1) implies (2).

Define the map b_1: $L(P) \to J$ by $b_1(S) = \bar{S}$. By the properties of a closure operator, it follows that b_1 is the right adjoint of i_K. By assumption (2), b_1 preserves arbitrary meets. By Remark 1.6, since $L(P)$ is a complete lattice, b_1 is dually residuated. Now, by Definition 2.7, $(J, L(P), i_K)$ is a continuous extension, and we have shown that (2) implies (3).

We have shown that b_1 is the right adjoint of i_K. If assumption (3) holds, then b_1 is dually residuated. Let c_1: $J \to L(P)$ be the right adjoint of b_1. Thus for each $A \in J$, $c_1(A) = b_1(c_1(A)) = A$. If $S \in L(P)$ and $A \subseteq \bar{S}$, then $A \subseteq b_1(S)$. Hence, by Lemma 1.9, $c_1(A) \subseteq S$. It follows that $\min\{S \in L(P) \mid A \subseteq \bar{S}\}$ exists and equals $c_1(A)$. This shows that (3) implies (4).

Define a map c_1: $J \to L(P)$ by putting $c_1(A) = \min\{S \in L(P) \mid A \subseteq b_1(S)\}$ for each $A \in J$. From this definition it follows that $b_1(c_1(A)) = A$ for each $A \in J$ and that $c_1(b_1(S)) \subseteq S$ for each $S \in L(P)$. Thus c_1 is the right adjoint of b_1. This shows that b_1 is dually residuated, and it follows that b_1 preserves arbitrary meets. Since b_1 is the right adjoint of i_K, b_1 is residuated and b_1 preserves arbitrary joins. Moreover, since i_K is one-one, b_1 is onto. Thus b_1 is a complete homomorphism from $L(P)$ onto J. Since $L(P)$ is a complete ring of sets, J is completely distributive. This proves that (4) implies (1).

5.7 COROLLARY. *For a chain-complete poset P the following three conditions are equivalent.*

(1) *P is a continuous poset.*

(2) *For every family $\{S_i \mid i \in I\} \subseteq L(P)$, $\bigcap \{\bar{S}_i \mid i \in I\} = \overline{\bigcap \{S_i \mid i \in I\}}$. (Here $\bar{S}_i$ denotes the Scott closure of S_i.)*

(3) *For every Scott closed set A, $\min\{S \in L(P) \mid A \subseteq \bar{S}\}$ exists.*

PROOF. By Lawson's Theorem (Theorem 4.8), if P is a chain-complete poset, then P is a continuous poset if and only if J $(= J_{\mathcal{D}}(P))$ is a completely distributive complete lattice. The rest follows from Proposition 5.6.

5.8 LEMMA. *If (P, Q, a) is a continuous extension and $A \in J$, then $R(A) = \min\{S \in L(P) \mid K(S) = A\}$.*

PROOF. The proof is based on the following observation: Let P be a poset and let $M \subseteq N \subseteq P$. If $x = \min(N)$ and $x \in M$, then $x = \min(M)$.

Let $M = \{S \in L(P) \mid K(S) = A\}$ and $N = \{S \in L(P) \mid A \subseteq K(S)\}$. By Lemma 4.11(6), (K, R) is a Galois connection from $L(P)$ to $L(P)$. Thus, by Lemma 1.9 and Definition 1.7, $R(A) = \min\{S \in L(P) \mid A \subseteq K(S)\} = \min(N)$. Since $A \in J$, it follows from Lemma 4.11(8) that $K(A) = A$. Thus, by Lemma 1.12, $A = K(A) = K(R(K(A))) = K(R(A))$. It follows that $R(A) \in M$, and thus we conclude that $R(A) = \min(M)$.

5.9 COROLLARY. *If (P, Q, a) is a strongly continuous extension and $A \in J$, then $R(A) = \min\{S \in L(P) \mid \bar{S} = A\}$.*

PROOF. By Theorem 4.18(6), for every $S \in L(P)$, $K(S) = \bar{S}$. The rest follows from Lemma 5.8.

5.10 COROLLARY. *If P is a $\mathfrak{D}$-continuous poset and $A \subseteq P$ is Scott closed, then $\{y \mid y \ll_{\mathfrak{D}} x$ for some $x \in A\}$ is the least decreasing set whose Scott closure is A.*

PROOF. By Lemma 4.21 every $\mathfrak{D}$-continuous poset is strongly $\mathfrak{D}$-continuous. The rest follows from Corollary 5.9.

6. Algebraic extensions. An important subclass of the class of strongly continuous extensions is the class of algebraic extensions.

6.1 DEFINITION. A continuous extension (P, Q, a) is called an *algebraic extension* if for each $x \in P$

$$\left\{y \mid y \underset{a}{\ll} x\right\} = \downarrow\left\{z \mid z \underset{a}{\ll} z \leqslant x\right\}.$$

Thus, in particular, if P is M-complete, then $(P, I_M(P), a_M)$ is an algebraic extension if and only if the following two conditions hold.

1. For each $x \in P$, $\downarrow\{z \mid z \ll_M z \leqslant x\} \in I_M(P)$.
2. For each $x \in P$, $x = \bigvee\{z \mid z \ll_M z \leqslant x\}$.

6.2 DEFINITION. Let (P, Q, a) be a continuous extension and let $s \in P$. If $s \ll_a s$, then s is called a *compact element*.

6.3 PROPOSITION. *Let (P, Q, a) be an algebraic extension and let $C = \{s \in P \mid s \ll_a s\}$ be the set of all compact elements of P. Then Γ $(= \Gamma^{Q,a})$ is order isomorphic to $L(C)$.*

PROOF. Define the map T_1: $L(C) \to \Gamma$ by $T_1(S) = \{a \in P \mid a \leqslant s$ for some $s \in S\}$, for each $S \in L(C)$. Next define the map T_2: $\Gamma \to L(C)$ by $T_2(B) = B \cap C$ for each $B \in \Gamma$. It is easy to see that T_1 and T_2 are well defined and order preserving. We will show that T_2 is the inverse of T_1.

Let $S \in L(C)$. Since $S \subseteq T_1(S)$ we obtain that $S \subseteq T_1(S) \cap C = T_2(T_1(S))$. Conversely, let $t \in T_2(T_1(S))$. Thus $t \in T_1(S)$ and $t \in C$. This means that $t \in C$ and that $t \leqslant s$ for some $s \in S$. Since $S \in L(C)$, it follows that $t \in S$.

Next let $B \in \Gamma$. Since $T_2(B) = B \cap C \subseteq B$, it follows that $\downarrow(B \cap C) \subseteq \downarrow B = B$ and thus $T_1(T_2(B)) \subseteq B$. Let $x \in B$; since $B \in \Gamma$, there exists $y \in B$ such that $x \ll_a y$. By assumption (P, Q, a) is an algebraic extension, and thus $x \ll_a s \ll_a s \ll_a y$ for some $s \in P$. We conclude that $s \in T_2(B)$ and thus $x \in T_1(T_2(B))$.

6.4 COROLLARY. *If (P, Q, a) is an algebraic extension then J $(= J^{Q,a})$ is a complete ring of sets.*

PROOF. It is well known (see **[13]**) that $L(C)$ is a complete ring of sets for any poset C. By Proposition 6.3, Γ is order isomorphic to $L(C)$. By Proposition 5.1, J is order isomorphic to Γ.

In a forthcoming article we investigate the algebraic extension $(P, I_M(P), a_M)$, where M is $\mathfrak{F}$, $\mathfrak{D}$ or $\mathfrak{P}$. Here we are proving only one result in this direction. From this theorem it follows that $(P, I_{\mathfrak{F}}(P), a_{\mathfrak{F}})$ is an algebraic extension if and only if it is a continuous extension.

6.5 THEOREM. *If P is a join semilattice, then the following conditions are equivalent.*
(1) *Every element of P can be expressed as a finite join of coprime elements.*
(2) *$J_{\mathfrak{F}}$ is a completely distributive complete lattice.*
(3) *$(P, I_{\mathfrak{F}}(P), a_{\mathfrak{F}})$ is an algebraic extension.*
(4) *$J_{\mathfrak{F}}$ is a complete ring of sets.*
(5) *$(P, I_{\mathfrak{F}}(P), a_{\mathfrak{F}})$ is a continuous extension.*

PROOF. J. Martinez proved in **[12]** that (1) is equivalent to (2). If P is a join semilattice, then, by Definition 2.1, P is $\mathfrak{F}$-complete. Assume that every element of P is a finite join of coprime elements and let $x \in P$. Let $A \subseteq P$ be a finite set of coprime elements such that $\bigvee A = x$. Let $y \in P$. If $y \ll_{\mathfrak{F}} x = \bigvee A$, then $y \leqslant a$ for some $a \in A$ and thus $\{y \mid y \ll_{\mathfrak{F}} x\} \subseteq \downarrow A$. The reverse inclusion follows from the fact that the elements of A are coprimes. Thus $\{y \mid y \ll_{\mathfrak{F}} x\} = \downarrow A$. This proves that $\{y \mid y \ll_{\mathfrak{F}} x\} \in I_{\mathfrak{F}}(P)$ and that $\bigvee\{y \mid y \ll_{\mathfrak{F}} x\} = x$. By Definition 2.3, P is $\mathfrak{F}$-continuous. We have just shown that for each $x \in P$ $\{y \mid \ll_{\mathfrak{F}} x\} = \downarrow\{z \mid z \ll_{\mathfrak{F}} z \leqslant x\}$, thus $(P, I_{\mathfrak{F}}(P), a_{\mathfrak{F}})$ is an algebraic extension. We have established that (1) implies (3).

By Corollary 6.4, (3) implies (4) and by Definition 6.1, (3) implies (5).

Every complete ring of sets is necessarily a completely distributive complete lattice, thus (4) implies (2). By Proposition 5.1(4), (5) implies (2).

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